A Relationship Between Dual-Intuitionistic Logic and Nelson's Constructive Logic (Sequent Calculi and Proof Theory)

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Abstract

Dual-intuitionistic logics are logics proposed by Czermak 1977, Goodman 1981 and Urbas 1996. We show a correspondence between Goodman's dual-intuitionistic logic and Nelson's constructive logic N^-. Moreover we introduce two types of cut-free sequent calculi for N^-.

1 Introduction

Dual-intuitionistic logics are logics proposed by Czermak 1977 [4], Goodman 1981 [6] and Urbas 1996 [12]. Czermak [4] introduces a Gentzen-type sequent calculus in which sequents have the restriction that whose antecedent contains at most one formula. This system is called by Czermak the "dual-intuitionistic calculus DJ". Goodman [6] introduces a logic which is called the "logic of contradiction" or the "anti-intuitionistic logic", and shows the completeness theorem (with respect to complete Brouwerian algebras) for this logic. Goodman's logic has a binary connective $\div$ named the "psudo-difference". We call here the logic, GJ and also call a $\div$-less its sublogic, GJ^- . Urbas [12] extends these logics of Czermak and Goodman, and moreover extends the cut-elimination result of Czermak, and discusses decidability, paraconsistency and so on. These logics containing Czermak's, Goodman's and Urbas's logics are called by Urbas the "dual-intuitionistic logics". Also an interesting related work for dual-intuitionistic logics is in Goré 2000 [5].

Logics with strong negation are first introduced by Nelson 1949 [9] and independently by Markov 1951. A constructive logic N^-, which is a variant of logics with strong negation, is posed by Almukdad and Nelson 1984 [1]. The paper [1] shows a relationship among constructive logics N^-, N^+ and Cleave's three-valued logic [3]. These logics such as N^-, N and N^+ have been studied by many logicians and computer scientists (e.g., [8, 10, 11, 13, 14]). For example, Wansing [14] provides some cut-free display calculi for N^- and N, which calculi have four-placed display sequents, and can derive the subformula property. In addition, Kamide [7] develops various kinds of cut-free sequent calculi for a intuitionistic linear logic with strong negation.

Logics with strong negation and dual-intuitionistic logics have been studied in the completely different fields and motivations. In this paper, we clarify a relationship among these different logics. Moreover, applying the resulting technique in [7] we propose new sequent calculi for N^-, and prove the cut-elimination theorems for these calculi.

This paper is organized as follows. In section 2, we present the dual-intuitionistic logics: Goodman's GJ and GJ^-, Czermak's DJ and Urbas's LDJ and $LDJ^{\div}$. In section 3, we introduce Nelson's constructive logic N^- and give a correspondence between GJ^- and N^-. Using this correspondence result, the cut-elimination theorem for GJ^- is proved. In section 4, we introduce a *dual calculus*

DC for N^-, and prove the cut-elimination theorem for DC. This calculus has two kinds of sequents: a *positive sequent* $\Gamma \Rightarrow^+ \gamma$ and a *negative sequent* $\Gamma \Rightarrow^- \gamma$. This idea of using two sorts of sequents is from Kripke-type semantics for logics with strong negation [11], which semantics have two types of valuations: $\models^+$ (corresponds to provability) and $\models^-$ (corresponds to refutability). We then show that GJ^- corresponds to the negative part (using only negative sequents) of a fragment of DC. In section 5, we introduce a *subformula calculus* SC for N^-, and prove the cut-elimination theorem and the subformula property for SC [1].

We assume here that the language of the negation-less fragment of the first-order predicate LJ consists of logical connectives $\wedge$, $\vee$ and $\rightarrow$ and quantifiers $\forall$ and $\exists$. Moreover we sometimes add the following connectives to the language: $\top$ ("sentential constant" for GJ), $\div$ ("psudo-difference" for some dual-intuitionistic logics) and $\neg$ ("negation" for some dual-intuitionistic logics) and $\sim$ ("strong negation" for N^-). Lower case Greek letters $\alpha, \beta, \ldots$ are used for formulas, and Greek capital letters $\Gamma, \Delta, \ldots$ are used for finite (possibly empty) sequences of formulas. $\sim \Gamma$ denotes the sequence $\langle \sim \gamma | \gamma \in \Gamma \rangle$. A *sequent* is an expression of the form $\Gamma \Rightarrow \gamma$ for N^- or $\gamma \Rightarrow \Gamma$ for dual-intuitionistic logics. Since all logics discussed in this paper are formulated as sequent calculi, we will sometimes identify a sequent calculus with the logic determined by it.

2 Dual-intuitionistic logics

First, we give a precise definition of Goodman's logic GJ [6]. In the following definitions, γ in expression $\gamma \Rightarrow \Gamma$ for any Γ means single formula.

Initial sequents of GJ are of the forms:

$$\alpha \Rightarrow \alpha, \qquad \gamma \Rightarrow \top.$$

The cut rule is of the form:

$$\frac{\gamma \Rightarrow \Delta, \alpha \quad \alpha \Rightarrow \Sigma}{\gamma \Rightarrow \Delta, \Sigma} \text{ (cut-d)}.$$

The inference rules of GJ [2] are as follows:

$$\frac{\gamma \Rightarrow \Delta, \alpha, \alpha}{\gamma \Rightarrow \Delta, \alpha} \text{ (co-d)}, \quad \frac{\gamma \Rightarrow \Delta}{\gamma \Rightarrow \Delta, \alpha} \text{ (we-d)}, \quad \frac{\gamma \Rightarrow \Delta, \beta, \alpha, \Gamma}{\gamma \Rightarrow \Delta, \alpha, \beta, \Gamma} \text{ (ex-d)},$$

$$\frac{\alpha \Rightarrow \Delta}{\alpha \wedge \beta \Rightarrow \Delta} \text{ ($\wedge$left1-d)}, \qquad \frac{\beta \Rightarrow \Delta}{\alpha \wedge \beta \Rightarrow \Delta} \text{ ($\wedge$left2-d)},$$

$$\frac{\gamma \Rightarrow \Delta, \alpha \quad \gamma \Rightarrow \Delta, \beta}{\gamma \Rightarrow \Delta, \alpha \wedge \beta} \text{ ($\wedge$right-d)}, \qquad \frac{\alpha \Rightarrow \Delta \quad \beta \Rightarrow \Delta}{\alpha \vee \beta \Rightarrow \Delta} \text{ ($\vee$left-d)},$$

$$\frac{\gamma \Rightarrow \Delta, \alpha}{\gamma \Rightarrow \Delta, \alpha \vee \beta} \text{ ($\vee$right1-d)}, \qquad \frac{\gamma \Rightarrow \Delta, \beta}{\gamma \Rightarrow \Delta, \alpha \vee \beta} \text{ ($\vee$right2-d)},$$

$$\frac{\alpha[t/x] \Rightarrow \Gamma}{\forall x \alpha \Rightarrow \Gamma} \text{ ($\forall$left-d)}, \qquad \frac{\gamma \Rightarrow \Gamma, \alpha[z/x]}{\gamma \Rightarrow \Gamma, \forall x \alpha} \text{ ($\forall$right-d)},$$

$$\frac{\alpha[z/x] \Rightarrow \Gamma}{\exists x \alpha \Rightarrow \Gamma} \text{ ($\exists$left-d)}, \qquad \frac{\gamma \Rightarrow \Gamma, \alpha[t/x]}{\gamma \Rightarrow \Gamma, \exists x \alpha} \text{ ($\exists$right-d)},$$

$$\frac{\alpha \Rightarrow \Delta, \beta}{\alpha \div \beta \Rightarrow \Delta} \text{ ($\div$left)}, \qquad \frac{\alpha \div \beta \Rightarrow \Delta}{\alpha \Rightarrow \Delta, \beta} \text{ ($\div$left^{-1})}.$$

Here, $\alpha[z/x]$ ($\alpha[t/x]$) is the formula obtained from α by replacing all free occurrences of x in α by an individual variable z (a term t, respectively), but avoiding the crash of variables. Also, in the

[1]The idea of showing subformula property was first appeared in [14].

[2]Strictly speaking, the original logic of Goodman has the quantifier rules which are slightly different forms.

rules for quantifiers, t is an arbitrary term and z is an arbitrary individual variable not occurring in the lower sequent.

Here we define $\mathrm{GJ}^- = \mathrm{GJ} - (\div\text{left}) - (\div\text{left}^{-1}) - (\gamma \Rightarrow \top)$.

In the following, we consider an expression $\gamma \Rightarrow \Gamma$ for any γ where γ denotes a single formula or an empty sequence.

Czermak's logic DJ [4] is defined as follows: DJ = GJ^- − (∃left-d) − (∃left-d) + (we-left-d) + (¬left-d) + (¬right-d) where

$$\frac{\Rightarrow \Delta}{\alpha \Rightarrow \Delta}\ (\text{we-left-d}), \qquad \frac{\Rightarrow \Delta, \alpha}{\neg\alpha \Rightarrow \Delta}\ (\neg\text{left-d}), \qquad \frac{\alpha \Rightarrow \Delta}{\Rightarrow \Delta, \neg\alpha}\ (\neg\text{right-d}).$$

Urbas's logics LDJ and $\mathrm{LDJ}^{\div}$ [12] are defined as follows: LDJ = DJ + (∃left-d) + (∃right-d) + (→left-d) + (→right1-d) + (→right2-d) and $\mathrm{LDJ}^{\div}$ = LDJ + (÷left) + (÷right) where

$$\frac{\gamma \Rightarrow \Delta, \alpha \quad \beta \Rightarrow \Gamma}{\gamma \Rightarrow \Delta, \Gamma, \alpha \div \beta}\ (\div\text{right}), \qquad \frac{\Rightarrow \Delta, \alpha \quad \beta \Rightarrow \Gamma}{\alpha \to \beta \Rightarrow \Delta, \Gamma}\ (\to\text{left-d}),$$

$$\frac{\alpha \Rightarrow \Delta}{\Rightarrow \Delta, \alpha \to \beta}\ (\to\text{right1-d}), \qquad \frac{\gamma \Rightarrow \Delta, \beta}{\gamma \Rightarrow \Delta, \alpha \to \beta}\ (\to\text{right2-d}).$$

3 Nelson's N^- and Goodman's GJ^-

We introduce Nelson's constructive logic N^-. In the following definitions, γ in expression $\Gamma \Rightarrow \gamma$ for any Γ means single formula.

Initial sequents of N^- are of the form:

$$\alpha \Rightarrow \alpha.$$

The cut rule of N^- is of the form:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \gamma}{\Gamma, \Sigma \Rightarrow \gamma}\ (\text{cut}).$$

The inference rules of $\sim$-free part of N^- are as follows:

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ (\text{co-left}), \qquad \frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ (\text{we-left}), \qquad \frac{\Delta, \beta, \alpha, \Gamma \Rightarrow \gamma}{\Delta, \alpha, \beta, \Gamma \Rightarrow \gamma}\ (\text{ex-left}),$$

$$\frac{\alpha, \Delta \Rightarrow \gamma}{\alpha \wedge \beta, \Delta \Rightarrow \gamma}\ (\wedge\text{left1}), \qquad \frac{\beta, \Delta \Rightarrow \gamma}{\alpha \wedge \beta, \Delta \Rightarrow \gamma}\ (\wedge\text{left2}),$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge\text{right}), \qquad \frac{\alpha, \Delta \Rightarrow \gamma \quad \beta, \Delta \Rightarrow \gamma}{\alpha \vee \beta, \Delta \Rightarrow \gamma}\ (\vee\text{left}),$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee\text{right1}), \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee\text{right2}),$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Sigma \Rightarrow \gamma}{\alpha \to \beta, \Gamma, \Sigma \Rightarrow \gamma}\ (\to\text{left}), \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\to\text{right}),$$

$$\frac{\alpha[t/x], \Gamma \Rightarrow \gamma}{\forall x\alpha, \Gamma \Rightarrow \gamma}\ (\forall\text{left}), \qquad \frac{\Gamma \Rightarrow \alpha[z/x]}{\Gamma \Rightarrow \forall x\alpha}\ (\forall\text{right}),$$

$$\frac{\alpha[z/x], \Gamma \Rightarrow \gamma}{\exists x\alpha, \Gamma \Rightarrow \gamma}\ (\exists\text{left}), \qquad \frac{\Gamma \Rightarrow \alpha[t/x]}{\Gamma \Rightarrow \exists x\alpha}\ (\exists\text{right}).$$

The strong negation inference rules of N^- are as follows:

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \sim\sim \alpha}\ (\sim\text{right}), \qquad \frac{\alpha, \Delta \Rightarrow \gamma}{\sim\sim \alpha, \Delta \Rightarrow \gamma}\ (\sim\text{left}),$$

$$\frac{\sim \alpha, \Delta \Rightarrow \gamma \quad \sim \beta, \Delta \Rightarrow \gamma}{\sim (\alpha \wedge \beta), \Delta \Rightarrow \gamma}\ (\sim \wedge\text{left}),$$

$$\frac{\Gamma \Rightarrow \sim \alpha}{\Gamma \Rightarrow \sim (\alpha \wedge \beta)}\ (\sim \wedge\text{right1}), \qquad \frac{\Gamma \Rightarrow \sim \beta}{\Gamma \Rightarrow \sim (\alpha \wedge \beta)}\ (\sim \wedge\text{right2}),$$

$$\frac{\sim \alpha, \Delta \Rightarrow \gamma}{\sim (\alpha \vee \beta), \Delta \Rightarrow \gamma}\ (\sim \vee\text{left1}), \qquad \frac{\sim \beta, \Delta \Rightarrow \gamma}{\sim (\alpha \vee \beta), \Delta \Rightarrow \gamma}\ (\sim \vee\text{left2}),$$

$$\frac{\Gamma \Rightarrow \sim \alpha \quad \Gamma \Rightarrow \sim \beta}{\Gamma \Rightarrow \sim (\alpha \vee \beta)}\ (\sim \vee\text{right}),$$

$$\frac{\sim \beta, \alpha, \Delta \Rightarrow \gamma}{\sim (\alpha \rightarrow \beta), \Delta \Rightarrow \gamma}\ (\sim \rightarrow\text{left}), \qquad \frac{\Gamma \Rightarrow \sim \beta \quad \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \sim (\alpha \rightarrow \beta)}\ (\sim \rightarrow\text{right}),$$

$$\frac{\sim \alpha[z/x], \Gamma \Rightarrow \gamma}{\sim \forall x\alpha, \Gamma \Rightarrow \gamma}\ (\sim \forall\text{left}), \qquad \frac{\Gamma \Rightarrow \sim \alpha[t/x]}{\Gamma \Rightarrow \sim \forall x\alpha}\ (\sim \forall\text{right}),$$

$$\frac{\sim \alpha[t/x], \Gamma \Rightarrow \gamma}{\sim \exists x\alpha, \Gamma \Rightarrow \gamma}\ (\sim \exists\text{left}), \qquad \frac{\Gamma \Rightarrow \sim \alpha[z/x]}{\Gamma \Rightarrow \sim \exists x\alpha}\ (\sim \exists\text{right}).$$

Here $\alpha[z/x]$ and $\alpha[t/x]$ denote the same meaning presented in section 2. We remark that this system is in [13, 10] and is different from that in [1], but both the systems are essentially the same thing. The system in [1] uses the multiple conclusion sequent, that is, the system is defined as so-called the "LJ′-style" formulation.

The following cut-elimination result is already presented in [10, 13].

Theorem 3.1 (Cut-Elimination Theorem for N^-) *The rule* (cut) *is admissible in cut-free* N^-.

The following theorem is a main result of this paper.

Theorem 3.2 (Correspondence Between GJ^- and N^-) *Let Γ be a sequence of $\sim$-free formulas, γ be a $\sim$-free formula.* (1) *$\gamma \Rightarrow \Gamma$ is provable in* GJ^- *if and only if $\sim \Gamma \Rightarrow \sim \gamma$ is provable in the $\rightarrow$-free fragment of* N^-. (2) *If $\sim \Gamma \Rightarrow \sim \gamma$ is cut-free provable in the $\rightarrow$-free fragment of* N^- *then $\gamma \Rightarrow \Gamma$ is cut-free provable in* GJ^-.

Using this theorem, we can obtain the following [3].

Theorem 3.3 (Cut-Elimination Theorem for GJ^-) *The rule* (cut-d) *is admissible in cut-free* GJ^-.

Proof Suppose that $\gamma \Rightarrow \Gamma$ is provable in GJ^-. Then $\sim \Gamma \Rightarrow \sim \gamma$ is provable in the $\rightarrow$-free fragment of N^- by Theorem 3.2 (1). By Theorem 3.1 and its subformula-property-like corollary [4], $\sim \Gamma \Rightarrow \sim \gamma$ is cut-free provable in the $\rightarrow$-free fragment of N^-. Hence $\gamma \Rightarrow \Gamma$ is cut-free provable in GJ^- by Theorem 3.2 (2). ∎

[3] Goodman [6] does not discuss the cut-elimination theorems for GJ and GJ^-. Maybe the cut-elimination theorem does not hold for GJ.

[4] We remark that N^- has no subformula property, but similar property holds for N^-.

4 Dual calculus for N^-

In this section, we introduce a dual calculus DC for N^-. In the following, a sequent of the form $\Gamma \Rightarrow^+ \gamma$ is calld a positive sequent, and a sequent of the form $\Gamma \Rightarrow^- \gamma$ is called a negative sequent. In the following definitions, γ in expression $\Gamma \Rightarrow^+ \gamma$ or $\Gamma \Rightarrow^- \gamma$ for any Γ means single formula.

The initial sequents of DC are of the forms:

$$\alpha \Rightarrow^+ \alpha, \qquad \alpha \Rightarrow^- \alpha.$$

The specific inference rules of DC are as follows:

$$\frac{\sim\Gamma, \Delta \Rightarrow^- \gamma}{\Gamma, \sim\Delta \Rightarrow^+ \sim\gamma}\ (-/+1), \qquad \frac{\sim\Gamma, \Delta \Rightarrow^- \sim\gamma}{\Gamma, \sim\Delta \Rightarrow^+ \gamma}\ (-/+2),$$

$$\frac{\sim\Gamma, \Delta \Rightarrow^+ \gamma}{\Gamma, \sim\Delta \Rightarrow^- \sim\gamma}\ (+/-1), \qquad \frac{\sim\Gamma, \Delta \Rightarrow^+ \sim\gamma}{\Gamma, \sim\Delta \Rightarrow^- \gamma}\ (+/-2).$$

The cut rules of DC are as follows:

$$\frac{\Gamma \Rightarrow^+ \alpha \quad \alpha, \Sigma \Rightarrow^+ \gamma}{\Gamma, \Sigma \Rightarrow^+ \gamma}\ (+\text{cut}), \qquad \frac{\Gamma \Rightarrow^- \alpha \quad \alpha, \Sigma \Rightarrow^- \gamma}{\Gamma, \Sigma \Rightarrow^- \gamma}\ (-\text{cut}).$$

The positive inference rules of DC are the same as that of $\sim$-free N^- where we use $\Rightarrow^+$ instead of $\Rightarrow$.

The negative inference rules of DC are as follows:

$$\frac{\Gamma \Rightarrow^- \gamma}{\alpha, \Gamma \Rightarrow^- \gamma}\ (-\text{we}), \qquad \frac{\alpha, \alpha, \Gamma \Rightarrow^- \gamma}{\alpha, \Gamma \Rightarrow^- \gamma}\ (-\text{co}), \qquad \frac{\Delta, \beta, \alpha, \Gamma \Rightarrow^- \gamma}{\Delta, \alpha, \beta, \Gamma \Rightarrow^- \gamma}\ (-\text{ex}),$$

$$\frac{\alpha, \Delta \Rightarrow^- \gamma \quad \beta, \Delta \Rightarrow^- \gamma}{\alpha \wedge \beta, \Delta \Rightarrow^- \gamma}\ (-\wedge\,\text{left}),$$

$$\frac{\Gamma \Rightarrow^- \alpha}{\Gamma \Rightarrow^- \alpha \wedge \beta}\ (-\wedge\,\text{right1}), \qquad \frac{\Gamma \Rightarrow^- \beta}{\Gamma \Rightarrow^- \alpha \wedge \beta}\ (-\wedge\,\text{right2}),$$

$$\frac{\alpha, \Delta \Rightarrow^- \gamma}{\alpha \vee \beta, \Delta \Rightarrow^- \gamma}\ (-\vee\,\text{left1}), \qquad \frac{\beta, \Delta \Rightarrow^- \gamma}{\alpha \vee \beta, \Delta \Rightarrow^- \gamma}\ (-\vee\,\text{left2}),$$

$$\frac{\Gamma \Rightarrow^- \alpha \quad \Gamma \Rightarrow^- \beta}{\Gamma \Rightarrow^- \alpha \vee \beta}\ (-\vee\,\text{right}),$$

$$\frac{\beta, \sim\alpha, \Delta \Rightarrow^- \gamma}{\alpha \rightarrow \beta, \Delta \Rightarrow^- \gamma}\ (-\!\rightarrow\!\text{left}), \qquad \frac{\Gamma \Rightarrow^- \beta \quad \Delta \Rightarrow^- \sim\alpha}{\Gamma, \Delta \Rightarrow^- \alpha \rightarrow \beta}\ (-\!\rightarrow\!\text{right}),$$

$$\frac{\alpha[z/x], \Gamma \Rightarrow^- \gamma}{\forall x\alpha, \Gamma \Rightarrow^- \gamma}\ (-\forall\text{left}), \qquad \frac{\Gamma \Rightarrow^- \alpha[t/x]}{\Gamma \Rightarrow^- \forall x\alpha}\ (-\forall\text{right}),$$

$$\frac{\alpha[t/x], \Gamma \Rightarrow^- \gamma}{\exists x\alpha, \Gamma \Rightarrow^- \gamma}\ (-\exists\text{left}), \qquad \frac{\Gamma \Rightarrow^- \alpha[z/x]}{\Gamma \Rightarrow^- \exists x\alpha}\ (-\exists\text{right}).$$

Here $\alpha[z/x]$ and $\alpha[t/x]$ denote the same meaning presented in section 2.

Theorem 4.1 (Equivalence Between DC **and** N^-**)** *Let Γ be a sequence of formulas, γ be a formula.* (1) *If $\Gamma \Rightarrow^* \gamma$ ($* \in \{+,-\}$) is provable in* DC*, then the sequent $\Gamma \Rightarrow \gamma$ is provable in N^- if $* = +$, or the sequent $\sim\Gamma \Rightarrow \sim\gamma$ is provable in N^- if $* = -$.* (2) *If $\Gamma \Rightarrow \gamma$ is cut-free provable in N^- then the sequent $\Gamma \Rightarrow^+ \gamma$ is cut-free provable in* DC.

Theorem 4.2 (Cut-Elimination Theorem for DC) *The rules* (+cut) *and* (−cut) *are admissible in cut-free* DC.

Proof Suppose that a sequent $\Gamma \Rightarrow^* \gamma$ ($* \in \{+,-\}$) is provable in DC. Then, by Theorem 4.1 (1), the sequent $\Gamma \Rightarrow \gamma$ is provable in N^- if $* = +$, or $\sim\Gamma \Rightarrow \sim\gamma$ is provable in N^- if $* = -$. Hence the sequent $\Gamma \Rightarrow \gamma$ or $\sim\Gamma \Rightarrow\sim\gamma$ is cut-free provable in N^- by Theorem 3.1. If $\Gamma \Rightarrow \gamma$ is cut-free provable in N^- then $\Gamma \Rightarrow^+ \gamma$ is cut-free provable in DC by Theorem 4.1 (2). If $\sim\Gamma \Rightarrow\sim\gamma$ is cut-free provable in N^- then $\sim\Gamma \Rightarrow^+\sim\gamma$ is cut-free provable in DC by Theorem 4.1 (2), and hence $\Gamma \Rightarrow^- \gamma$ is cut-free provable in DC. ∎

Theorem 4.3 (Correspondence Between GJ^- **and** DC) *Let* Γ *be a sequence of* $\sim$*-free formulas,* γ *be a* $\sim$*-free formula.* $\gamma \Rightarrow \Gamma$ *is provable in* GJ^- *if and only if* $\Gamma \Rightarrow^- \gamma$ *is provable in the* $\rightarrow$*-free fragment of* DC.

In the following, we assume the language of N by deleting $\exists$. Let GBL be the predicate version of Arieli and Avron's logic [2] which is called the "logic of logical bilattices" [5], and N_r^- and N_r^+ be the multiple conclusion sequent calculi for the $\rightarrow$-free parts of Nelson's logics N^- and N^+ [6] presented in [1], and C be Cleave's logic [3] [7]. We can conclude the following fact:

$$GJ^- \subseteq^- \rightarrow\text{-free-DC} \doteq \rightarrow\text{-free-}N^- \subseteq N_r^- \subseteq \otimes\oplus\text{-free-GBL} \subseteq C \subseteq N_r^+$$

where $\subseteq^-$ denotes the result of Theorem 4.3, $\doteq$ denotes the results of Theorems 3.1, 4.1 and 4.2, and $\subseteq$ denotes the inclusion between the sets of provable sequents. We remark that the fact $N_r^- \subseteq C \subseteq N_r^+$ above is established by Almukdad and Nelson [1].

5 Subformula calculus for N^-

In this section, we introduce a subformula calculus SC for N^-. The sequent of SC is of the forms $\Gamma : \Delta \Rightarrow \emptyset : \gamma$ and $\Gamma : \Delta \Rightarrow \gamma : \emptyset$ where γ is a formula, and Γ and Δ are sequences of formulas. The sequents

$$\gamma_1, ..., \gamma_m : \delta_1, ..., \delta_n \Rightarrow \emptyset : \gamma, \qquad \gamma_1, ..., \gamma_m : \delta_1, ..., \delta_n \Rightarrow \gamma : \emptyset$$

($0 \leq m, n$) in SC intuitively mean that

$$\sim\gamma_1, ..., \sim\gamma_m, \delta_1, ..., \delta_n \Rightarrow \gamma, \qquad \sim\gamma_1, ..., \sim\gamma_m, \delta_1, ..., \delta_n \Rightarrow\sim\gamma$$

in N^-. In the following definitions, C means $\emptyset : \gamma$ or $\gamma : \emptyset$.

The initial sequents of SC are of the forms:

$$\emptyset : \alpha \Rightarrow \emptyset : \alpha, \qquad \alpha : \emptyset \Rightarrow \alpha : \emptyset.$$

The specific inference rules of SC are as follows:

$$\frac{\Gamma : \Delta \Rightarrow \alpha : \emptyset}{\Gamma : \Delta \Rightarrow \emptyset :\sim\alpha}\ (\sim\text{r}+), \qquad \frac{\Gamma : \Delta \Rightarrow \emptyset : \alpha}{\Gamma : \Delta \Rightarrow\sim\alpha : \emptyset}\ (\sim\text{r}-),$$

$$\frac{\alpha, \Gamma : \Delta \Rightarrow C}{\Gamma :\sim\alpha, \Delta \Rightarrow C}\ (\sim\text{l}+), \qquad \frac{\Gamma : \alpha, \Delta \Rightarrow C}{\sim\alpha, \Gamma : \Delta \Rightarrow C}\ (\sim\text{l}-).$$

[5] Strictly speaking, the original GBL is a propositional logic, but section 3.5 in [2] proposes the quantifier rules for GBL. We consider here the predicate extension. GBL has fusion and fission connectives $\otimes$ and $\oplus$. Also GBL is regarded as a logic with strong negation.

[6] N^+ is first introduced by Thomason [11]. This logic is obtained from (a Hilbert-style system) N by adding the constant domain axiom scheme: $\forall x(\alpha(x) \vee \beta) \rightarrow \forall \alpha(x) \vee \beta$ where x is not free in β. N is obtained from (a Hilbert-style system) N^- by adding the axiom scheme: $\alpha \wedge \sim\alpha \rightarrow \beta$.

[7] The original logic of Cleave has no structural rules, and hence we must add the appropriate structural rules. The logic with the structural rules is called C in the present paper.

The cut rules of SC are as follows:

$$\frac{\Gamma_1 : \Delta_1 \Rightarrow \alpha : \emptyset \quad \alpha, \Gamma_2 : \Delta_2 \Rightarrow C}{\Gamma_1, \Gamma_2 : \Delta_1, \Delta_2 \Rightarrow C} \text{ (cut}-\text{)},$$

$$\frac{\Gamma_1 : \Delta_1 \Rightarrow \emptyset : \alpha \quad \Gamma_2 : \alpha, \Delta_2 \Rightarrow C}{\Gamma_1, \Gamma_2 : \Delta_1, \Delta_2 \Rightarrow C} \text{ (cut+)}.$$

The positive inference rules of SC are as follows:

$$\frac{\Gamma : \Delta \Rightarrow C}{\Gamma : \alpha, \Delta \Rightarrow C} \text{ (w+)}, \quad \frac{\Gamma : \alpha, \alpha, \Delta \Rightarrow C}{\Gamma : \alpha, \Delta \Rightarrow C} \text{ (c+)}, \quad \frac{\Gamma : \Delta_1, \beta, \alpha, \Delta_2 \Rightarrow C}{\Gamma : \Delta_1, \alpha, \beta, \Delta_2 \Rightarrow C} \text{ (e+)},$$

$$\frac{\Gamma_1 : \Delta_1 \Rightarrow \emptyset : \alpha \quad \Gamma_2 : \beta, \Delta_2 \Rightarrow C}{\Gamma_1, \Gamma_2 : \alpha \rightarrow \beta, \Delta_1, \Delta_2 \Rightarrow C} \text{ (}\rightarrow\text{l+)}, \quad \frac{\Gamma : \alpha, \Delta \Rightarrow \emptyset : \beta}{\Gamma : \Delta \Rightarrow \emptyset : \alpha \rightarrow \beta} \text{ (}\rightarrow\text{r+)},$$

$$\frac{\Gamma : \alpha, \Delta \Rightarrow C}{\Gamma : \alpha \wedge \beta, \Delta \Rightarrow C} \text{ (}\wedge\text{l1+)}, \quad \frac{\Gamma : \beta, \Delta \Rightarrow C}{\Gamma : \alpha \wedge \beta, \Delta \Rightarrow C} \text{ (}\wedge\text{l2+)},$$

$$\frac{\Gamma : \Delta \Rightarrow \emptyset : \alpha \quad \Gamma : \Delta \Rightarrow \emptyset : \beta}{\Gamma : \Delta \Rightarrow \emptyset : \alpha \wedge \beta} \text{ (}\wedge\text{r1+)}, \quad \frac{\Gamma : \alpha, \Delta \Rightarrow C \quad \Gamma : \beta, \Delta \Rightarrow C}{\Gamma : \alpha \vee \beta, \Delta \Rightarrow C} \text{ (}\vee\text{l1+)},$$

$$\frac{\Gamma : \Delta \Rightarrow \emptyset : \alpha}{\Gamma : \Delta \Rightarrow \emptyset : \alpha \vee \beta} \text{ (}\vee\text{r1+)}, \quad \frac{\Gamma : \Delta \Rightarrow \emptyset : \beta}{\Gamma : \Delta \Rightarrow \emptyset : \alpha \vee \beta} \text{ (}\vee\text{r2+)},$$

$$\frac{\Gamma : \alpha[t/x], \Delta \Rightarrow C}{\Gamma : \forall x\alpha, \Delta \Rightarrow C} \text{ (}\forall\text{l+)}, \quad \frac{\Gamma : \Delta \Rightarrow \emptyset : \alpha[z/x]}{\Gamma : \Delta \Rightarrow \emptyset : \forall x\alpha} \text{ (}\forall\text{r+)},$$

$$\frac{\Gamma : \alpha[z/x], \Delta \Rightarrow C}{\Gamma : \exists x\alpha, \Delta \Rightarrow C} \text{ (}\exists\text{l+)}, \quad \frac{\Gamma : \Delta \Rightarrow \emptyset : \alpha[t/x]}{\Gamma : \Delta \Rightarrow \emptyset : \exists x\alpha} \text{ (}\exists\text{r+)}.$$

The negative inference rules of SC are as follows:

$$\frac{\Gamma : \Delta \Rightarrow C}{\alpha, \Gamma : \Delta \Rightarrow C} \text{ (w}-\text{)}, \quad \frac{\alpha, \alpha, \Gamma : \Delta \Rightarrow C}{\alpha, \Gamma : \Delta \Rightarrow C} \text{ (c}-\text{)}, \quad \frac{\Gamma_1, \beta, \alpha, \Gamma_2 : \Delta \Rightarrow C}{\Gamma_1, \alpha, \beta, \Gamma_2 : \Delta \Rightarrow C} \text{ (e}-\text{)},$$

$$\frac{\beta, \Gamma : \alpha, \Delta \Rightarrow C}{\alpha \rightarrow \beta, \Gamma : \Delta \Rightarrow C} \text{ (}\rightarrow\text{l}-\text{)}, \quad \frac{\Gamma_1 : \Delta_1 \Rightarrow \beta : \emptyset \quad \Gamma_2 : \Delta_2 \Rightarrow \emptyset : \alpha}{\Gamma_1, \Gamma_2 : \Delta_1, \Delta_2 \Rightarrow \alpha \rightarrow \beta : \emptyset} \text{ (}\rightarrow\text{r}-\text{)},$$

$$\frac{\alpha, \Gamma : \Delta \Rightarrow C \quad \beta, \Gamma : \Delta \Rightarrow C}{\alpha \wedge \beta, \Gamma : \Delta \Rightarrow C} \text{ (}\wedge\text{l}-\text{)},$$

$$\frac{\Gamma : \Delta \Rightarrow \alpha : \emptyset}{\Gamma : \Delta \Rightarrow \alpha \wedge \beta : \emptyset} \text{ (}\wedge\text{r1}-\text{)}, \quad \frac{\Gamma : \Delta \Rightarrow \beta : \emptyset}{\Gamma : \Delta \Rightarrow \alpha \wedge \beta : \emptyset} \text{ (}\wedge\text{r2}-\text{)},$$

$$\frac{\alpha, \Gamma : \Delta \Rightarrow C}{\alpha \vee \beta, \Gamma : \Delta \Rightarrow C} \text{ (}\vee\text{l1}-\text{)}, \quad \frac{\beta, \Gamma : \Delta \Rightarrow C}{\alpha \vee \beta, \Gamma : \Delta \Rightarrow C} \text{ (}\vee\text{l2}-\text{)},$$

$$\frac{\Gamma : \Delta \Rightarrow \alpha : \emptyset \quad \Gamma : \Delta \Rightarrow \beta : \emptyset}{\Gamma : \Delta \Rightarrow \alpha \vee \beta : \emptyset} \text{ (}\vee\text{r}-\text{)},$$

$$\frac{\alpha[z/x], \Gamma : \Delta \Rightarrow C}{\forall x\alpha, \Gamma : \Delta \Rightarrow C} \text{ (}\forall\text{l}-\text{)}, \quad \frac{\Gamma : \Delta \Rightarrow \alpha[t/x] : \emptyset}{\Gamma : \Delta \Rightarrow \forall x\alpha : \emptyset} \text{ (}\forall\text{r}-\text{)},$$

$$\frac{\alpha[t/x], \Gamma : \Delta \Rightarrow C}{\exists x\alpha, \Gamma : \Delta \Rightarrow C} \text{ (}\exists\text{l}-\text{)}, \quad \frac{\Gamma : \Delta \Rightarrow \alpha[z/x] : \emptyset}{\Gamma : \Delta \Rightarrow \exists x\alpha : \emptyset} \text{ (}\exists\text{r}-\text{)}.$$

Here $\alpha[z/x]$ and $\alpha[t/x]$ denote the same meaning presented in section 2.

Theorem 5.1 (Equivalence Between SC **and** N^-) *Let* Γ *and* Δ *be sequences of formulas,* γ *be a formula.* (1) *If* $\Gamma : \Delta \Rightarrow C$ (C *is either* $\emptyset : \gamma$ *or* $\gamma : \emptyset$) *is provable in* SC, *then the sequent* $\sim \Gamma, \Delta \Rightarrow C'$ *is provable in* N^- *where* $C' \equiv \gamma$ *if* $C \equiv \emptyset : \gamma$ *and* $C' \equiv \sim \gamma$ *if* $C \equiv \gamma : \emptyset$. (2) *If* $\sim \Gamma, \Delta \Rightarrow C'$ (C' *is either* γ *or* $\sim \gamma$) *is cut-free provable in* N^-, *then the sequent* $\Gamma : \Delta \Rightarrow C$ *is cut-free provable in* SC *where* $C \equiv \emptyset : \gamma$ *if* $C' \equiv \gamma$ *and* $C \equiv \gamma : \emptyset$ *if* $C' \equiv \sim \gamma$.

Theorem 5.2 (Cut-Elimination Theorem for SC) *The rules* (cut+) *and* (cut−) *are admissible in cut-free* SC.

Proof Suppose that a sequent $\Gamma : \Delta \Rightarrow C$ is provable in SC. Then the sequent $\sim \Gamma, \Delta \Rightarrow C'$ is provable in N^- by Theorem 5.1 (1), and hence the sequent $\sim \Gamma, \Delta \Rightarrow C'$ is cut-free provable in N^- by Theorem 3.1. Therefere $\Gamma : \Delta \Rightarrow C$ is cut-free provable in SC by Theorem 5.1 (2). ∎

Corollary 5.3 (Subformula Property for SC) *The calculus* SC *has the subformula property, that is, if a sequent* S *is provable in* SC, *then there is a proof of* S *such that any formula appearing in it is a subformula of some formula in* S.

Theorem 5.4 (Correspondence Between GJ^- **and** SC) *Let* Γ *be a sequence of* $\sim$*-free formulas,* γ *be a* $\sim$*-free formula.* $\gamma \Rightarrow \Gamma$ *is provable in* GJ^- *if and only if* $\Gamma : \emptyset \Rightarrow \gamma : \emptyset$ *is provable in the* $\rightarrow$*-free fragment of* SC.

6 Notes

Russell's paradox in some naive set theories based on dual-intuitionistic logics was discussed in [6, 12]. In addition, the paper [12] remarks that Curry's paradox cannot be reproduced in these set theories. We mention, in the following, Russell's paradox over N^-.

Before the discussion for the case of N^-, we consider a derivation in LJ. Let $\alpha \equiv t \in t$ where $t \equiv \{x | \neg(x \in x)\}$ and assume that the sequents $\neg\alpha \Rightarrow \alpha$ and $\alpha \Rightarrow \neg\alpha$ are provable. Then we have the following derivation:

$$\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \neg\alpha \qquad \dfrac{\dfrac{\neg\alpha \Rightarrow \alpha}{\neg\alpha, \neg\alpha \Rightarrow}\,(\neg\text{left})}{\neg\alpha \Rightarrow}\,(\text{co-left})}{\alpha \Rightarrow}\,(\text{cut})}{\Rightarrow \neg\alpha}\,(\neg\text{right}) \qquad \dfrac{\dfrac{\neg\alpha \Rightarrow \alpha}{\neg\alpha, \neg\alpha \Rightarrow}\,(\neg\text{left})}{\neg\alpha \Rightarrow}\,(\text{co-left})}{\Rightarrow}\,(\text{cut}).$$

This means that Russell's paradox derives contradiction. It is well-known that, in this derivation, the applications of the contraction rule (co-left) are the causes of the contradiction. However, in this derivation, we feel that the applications of the rules (¬left) and (¬right) are the causes. Then, in order to avoid the contradiction, we use N^- as a basis of naive set theory. In other words, we adopt the strong negation $\sim$ instead of the usual (intuitionistic or classical) negation $\neg$. Here we remark that the language of N^- has no usual negation $\neg$ and falsum constant $\bot$. Hence we cannot define $\neg\alpha := \alpha \rightarrow \bot$.

We then have the following conjecture: the naive set theory (with unrestricted comprehension) based on N^- is consistent (i.e., Russell's paradox does not derive the fact that the empty sequent $\Rightarrow$ is provable).

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